5.5 (continued)

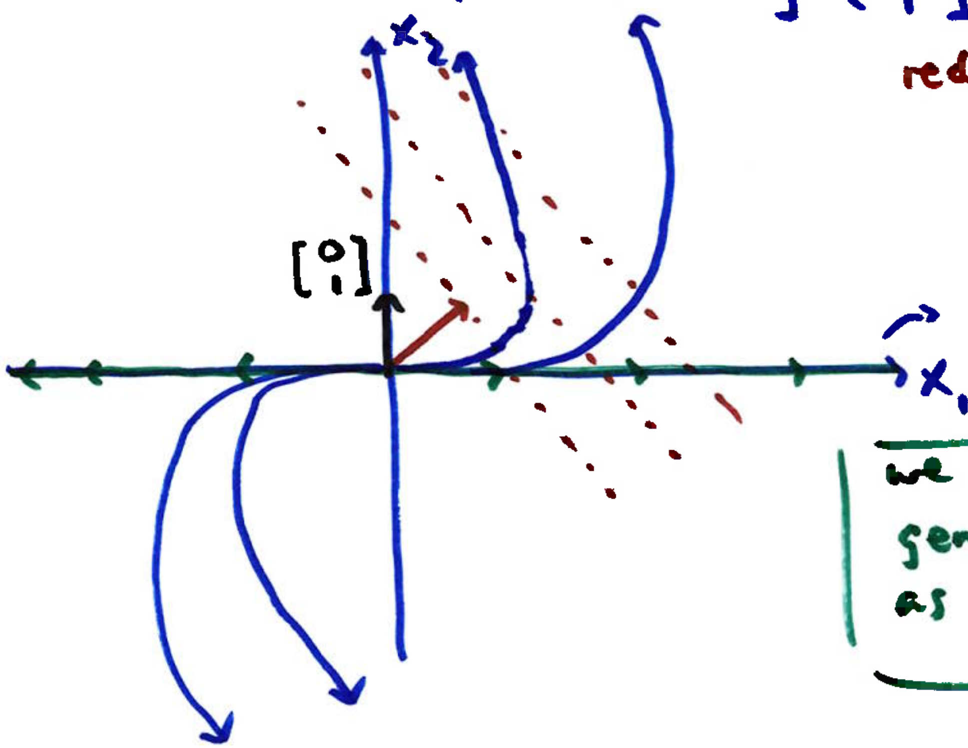
from last time: $\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}$

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

phase diagram: $\vec{x} \to \vec{0}$ as $t \to -\infty$ $\to$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ more important (asymptote)

as t increases, move away

solutions go in the direction of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ while being nudged by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



red vector: ~~stem~~ sum of true and generalized eigenvectors

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (true eigenvector)

we do not see the generalized eigenvector as an asymptote

3X3 system w/ λ repeating 3 times

possibilities: 1 : full set of eigenvectors

2 : missing one eigenvector (defect of one)

3 : " two " (" " two)

defect of two is actually simpler

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

$\lambda = 2, 2, 2$ $\qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ missing two

let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (true)

$(A-\lambda I)\vec{v}_2 = \vec{v}_1 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$\vdots$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solution 1: $e^{\lambda t}\vec{v}_1$

Solution 2: $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

Solution 3: $e^{\lambda t}\left(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$

general solution

$$\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\right) + c_3 e^{2t}\left(\frac{1}{2}t^2\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

<u>alternate method</u>:

$$(A-\lambda I)\vec{v}_1 = \vec{0}$$
$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$
$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

multiply last eq. by $(A-\lambda I)$

$$(A-\lambda I)^2\vec{v}_3 = (A-\lambda I)\vec{v}_2 = \vec{v}_1$$

again

$$(A-\lambda I)^3\vec{v}_3 = (A-\lambda I)\vec{v}_1 = \vec{0} \qquad (A-\lambda I)^3\vec{v}_3 = \vec{0}$$

not a coincidence: if defect is k then $(A-\lambda I)^{k+1} = \vec{0}$

also, $(A-\lambda I)^{k+1}$ is <u>always</u> a zero matrix

so, $(A-\lambda I)^3 \vec{v}_3 = \vec{0}$

becomes $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\vec{v}_3$ is arbitrary*

* $\vec{v}_3 \neq \vec{0}$ and must be linearly indp from true eigenvectors

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ e \\ \pi \end{bmatrix}$$

here, let's use $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

now rebuild the <u>entire</u> chain, including $\vec{v}_1$

$$(A-\lambda I)\vec{v}_3 = \vec{v}_2 = \dots = \begin{bmatrix} \cancel{0} \\ \cancel{1} \\ \cancel{0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1 = \dots = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ matches the true eigenvector here, but not always

then form solutions as before

two methods: 1st is "stepping up"

2nd is "stepping down"

for the case w/ defect of one (missing one eigenvector)

"stepping down" is much easier

$$\vec{x}' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \vec{x} \qquad \lambda = 1, 1, 1$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

defect is 1 so $\underbrace{(A - \lambda I)^{1+1}}_{\text{zero matrix}} \vec{v}_3 = \vec{0}$

$\vec{v}_3$ is arbitrary* $\quad \vec{v}_3 \neq \vec{0}$ and linearly indp from true eigenvectors here, let's use $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

step down : $(A-\lambda I)\vec{v}_3 = \vec{v}_2 = \cdots = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can't have $\vec{0}$

because $\vec{v}_2$ is a linear combination of the two true eigenvectors

when this happens, pick either of the true eigenvectors to be $\vec{v}_1$

here, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$\boxed{\begin{array}{ll} \text{Solution 1:} & e^{\lambda t}\vec{v}_1 \\ \text{" } \quad 2: & e^{\lambda t}\vec{v}_2 \\ \text{" } \quad 3: & e^{\lambda t}(t\vec{v}_2 + \vec{v}_3) \end{array}}$$